



v0.8 (Shintani & Hirokawa, JAIST)

confluence tool for **left-linear** TRSs, supporting:



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1 compositional confluence criteria

(FSCD 2022)



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Theorem (reduction method)

$CR(\mathcal{R}) \iff CR(\mathcal{C})$ if \mathcal{R} is left-linear, $\mathcal{C} \subseteq \mathcal{R}$, $PCP(\mathcal{R}) \subseteq \leftrightarrow_{\mathcal{C}}^*$, and $\mathcal{R}|_{\mathcal{C}} \subseteq \rightarrow_{\mathcal{C}}^*$

confluence tool for **left-linear** TRSs, supporting:

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Theorem (reduction method)

$CR(\mathcal{R}) \iff CR(\mathcal{C})$ if \mathcal{R} is left-linear, $\mathcal{C} \subseteq \mathcal{R}$, $PCP(\mathcal{R}) \subseteq \leftrightarrow_{\mathcal{C}}^*$, and $\mathcal{R}|_{\mathcal{C}} \subseteq \rightarrow_{\mathcal{C}}^*$
where $\mathcal{R}|_{\mathcal{C}} = \{\ell \rightarrow r \in \mathcal{R} \mid \mathcal{F}un(\ell) \subseteq \mathcal{F}un(\mathcal{C})\}$

using **reduction method**, we show confluence of left-linear TRS \mathcal{R} :

$$\begin{array}{lll} 1: & x + 0 \rightarrow x & 3: \quad 0 + y \rightarrow y & 5: \quad s(x) + y \rightarrow s(x + y) \\ 2: & x \times 0 \rightarrow 0 & 4: \quad s(x) \times 0 \rightarrow 0 & 6: \quad s(x) \times y \rightarrow (x \times y) + y \end{array}$$

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$$\boxed{1} \text{ CR}(\mathcal{R}) \iff \text{CR}(\{1, 2, 3\})$$

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1 $CR(\mathcal{R}) \iff CR(\{1, 2, 3\})$ because

$$\begin{array}{ccc} & 0 + 0 & \\ & \swarrow \neq \searrow & \\ 0 & & 0 \end{array}$$

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 \swarrow \neq & \swarrow \neq & \swarrow \neq & \swarrow \neq \\
 0 = 0 & 0 = 0 & s(x) & s(x + 0) & s(x + 0) & s(x) \\
 & & \xrightarrow{1} & & &
 \end{array}$$

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$$\begin{array}{cccc}
 \begin{array}{c} 0 + 0 \\ \swarrow \not\equiv \searrow \\ 0 = 0 \end{array} &
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 \begin{array}{c} s(x) + 0 \\ \swarrow \not\equiv \searrow \\ s(x) \quad s(x + 0) \\ \text{---} \text{---} \text{---} \\ \text{1} \end{array} &
 \begin{array}{c} s(x) + 0 \\ \swarrow \not\equiv \searrow \\ s(x + 0) \quad s(x) \\ \text{---} \text{---} \text{---} \\ \text{1} \end{array}
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$$\begin{array}{ccccc}
 0 + 0 & 0 \times 0 & s(x) + 0 & s(x) + 0 & s(x) \times 0 \\
 \swarrow \neq & \swarrow \neq & \swarrow \neq & \swarrow \neq & \swarrow \neq \\
 0 = 0 & 0 = 0 & s(x) & s(x + 0) & 0 \\
 & & \xrightarrow{1} & \xrightarrow{1} & \\
 & & s(x + 0) & s(x) & (x \times 0) + 0
 \end{array}$$

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$0 + 0$
 0×0
 $s(x) + 0$
 $s(x) + 0$
 $s(x) \times 0$

$\swarrow \neq \searrow$
 $\swarrow \neq \searrow$
 $\swarrow \neq \searrow$
 $\swarrow \neq \searrow$
 $\swarrow \neq \searrow$

$0 = 0$
 $0 = 0$
 $s(x)$
 $s(x + 0)$
 $s(x + 0)$
 $s(x)$
 0
 $(x \times 0) + 0$

$\xrightarrow{1}$
 $\xrightarrow{1}$
 $\xrightarrow{2} \cdot \xrightarrow{1}$

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$0 + 0$
 0×0
 $s(x) + 0$
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$\swarrow \not\equiv \searrow$
 $\swarrow \not\equiv \searrow$
 $\swarrow \not\equiv \searrow$
 $\swarrow \not\equiv \searrow$
 $\swarrow \not\equiv \searrow$
 $\swarrow \not\equiv \searrow$

$0 = 0$
 $0 = 0$
 $s(x) \quad s(x + 0)$
 $s(x + 0) \quad s(x)$
 $0 \quad (x \times 0) + 0$
 $(x \times 0) + 0 \quad 0$

\curvearrowright
 \curvearrowright
 \curvearrowright
 \curvearrowright
 \curvearrowright

1
 1
 2
 1

using **reduction method**, we show confluence of left-linear TRS \mathcal{R} :

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 1: & x + 0 \rightarrow x & 3: & 0 + y \rightarrow y & 5: & s(x) + y \rightarrow s(x + y) \\
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The diagram illustrates confluence proofs for three rules:

- Rule 1:** $0 + 0 \rightarrow 0$. Two paths from $0 + 0$ lead to $0 = 0$.
- Rule 2:** $0 \times 0 \rightarrow 0$. Two paths from 0×0 lead to $0 = 0$.
- Rule 3:** $s(x) + 0 \rightarrow s(x)$. Two paths from $s(x) + 0$ lead to $s(x)$ and $s(x + 0)$. A red arrow labeled '1' indicates confluence between these two results.
- Rule 4:** $s(x) \times 0 \rightarrow 0$. Two paths from $s(x) \times 0$ lead to 0 and $(x \times 0) + 0$. A red arrow labeled '1' indicates confluence between these two results.
- Rule 5:** $s(x) + y \rightarrow s(x + y)$. Two paths from $s(x) + y$ lead to 0 and $(x \times 0) + 0$. Red arrows labeled '2' and '1' indicate confluence between these two results.
- Rule 6:** $s(x) \times y \rightarrow (x \times y) + y$. Two paths from $s(x) \times 0$ lead to $(x \times 0) + 0$ and 0 . Red arrows labeled '1' and '2' indicate confluence between these two results.

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$$\begin{array}{c}
 0 + 0 \\
 \swarrow \searrow \\
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 \end{array}
 \quad
 \begin{array}{c}
 0 \times 0 \\
 \swarrow \searrow \\
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 \end{array}
 \quad
 \begin{array}{c}
 s(x) + 0 \\
 \swarrow \searrow \\
 s(x) \quad s(x + 0) \\
 \text{---} \text{---} \\
 \text{1}
 \end{array}
 \quad
 \begin{array}{c}
 s(x) + 0 \\
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 \text{---} \text{---} \\
 \text{1}
 \end{array}
 \quad
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 s(x) \times 0 \\
 \swarrow \searrow \\
 0 \quad (x \times 0) + 0 \\
 \text{---} \text{---} \\
 \text{2} \quad \text{1}
 \end{array}
 \quad
 \begin{array}{c}
 s(x) \times 0 \\
 \swarrow \searrow \\
 (x \times 0) + 0 \quad 0 \\
 \text{---} \text{---} \\
 \text{1} \quad \text{2}
 \end{array}$$

and subsystem over $\mathcal{F}un(\{1, 2\}) = \{0, +, \times\}$

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 \begin{array}{c} s(x) \times 0 \\ \swarrow \not\equiv \searrow \\ (x \times 0) + 0 \quad 0 \\ \xrightarrow{1} \quad \xrightarrow{2} \end{array}
 \end{array}$$

and subsystem over $\mathcal{F}un(\{1, 2\}) = \{0, +, \times\}$ is $\{1, 2, 3\}$

using **reduction method**, we show confluence of left-linear TRS \mathcal{R} :

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 \end{array}$$

1 $CR(\mathcal{R}) \iff CR(\{1, 2, 3\})$ because

$0 + 0$ and 0×0 both reduce to $0 = 0$.
 $s(x) + 0$ reduces to $s(x)$.
 $s(x) + 0$ also reduces to $s(x + 0)$.
 $s(x)$ and $s(x + 0)$ are connected by a red arrow labeled 1.
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 $s(x + 0)$ and $s(x)$ are connected by a red arrow labeled 1.
 $s(x) \times 0$ reduces to 0 .
 $s(x) \times 0$ also reduces to $(x \times 0) + 0$.
 0 and $(x \times 0) + 0$ are connected by a red arrow labeled 2.
 $(x \times 0) + 0$ reduces to 0 .
 $(x \times 0) + 0$ also reduces to 0 .
 0 and 0 are connected by a red arrow labeled 1.

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 \end{array}$$

1 $CR(\mathcal{R}) \iff CR(\{1, 2, 3\})$ because

$0 + 0 \rightarrow 0 = 0$ (no confluence arrows)
 $0 \times 0 \rightarrow 0 = 0$ (no confluence arrows)
 $s(x) + 0 \rightarrow s(x) \rightarrow s(x + 0)$ (confluence arrow labeled 1)
 $s(x) + 0 \rightarrow s(x + 0) \rightarrow s(x)$ (confluence arrow labeled 1)
 $s(x) \times 0 \rightarrow 0 \rightarrow (x \times 0) + 0$ (confluence arrows labeled 2 and 1)
 $s(x) \times 0 \rightarrow (x \times 0) + 0 \rightarrow 0$ (confluence arrows labeled 1 and 2)

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$0 + 0$ and 0×0 both reduce to $0 = 0$.
 $s(x) + 0$ and $s(x) \times 0$ both reduce to $s(x)$ (via rule 1) and $s(x + 0)$ (via rule 3).
 $s(x) + 0$ and $s(x) \times 0$ both reduce to $s(x + 0)$ (via rule 1) and $s(x)$ (via rule 3).
 $s(x) \times 0$ and $(x \times 0) + 0$ both reduce to 0 (via rule 2) and $(x \times 0) + 0$ (via rule 6).
 $(x \times 0) + 0$ and 0 both reduce to 0 (via rule 2).

and subsystem over $\mathcal{F}un(\{1, 2\}) = \{0, +, \times\}$ is $\{1, 2, 3\}$

2 $CR(\{1, 2, 3\}) \iff CR(\emptyset)$

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$\begin{array}{c}
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 \swarrow \searrow \\
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 \end{array}
 \quad
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 \swarrow \searrow \\
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 \end{array}
 \quad
 \begin{array}{c}
 s(x) + 0 \\
 \swarrow \searrow \\
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 \quad \quad \quad \curvearrowright \\
 \quad \quad \quad 1
 \end{array}
 \quad
 \begin{array}{c}
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 \swarrow \searrow \\
 s(x + 0) \quad s(x) \\
 \quad \quad \quad \curvearrowright \\
 \quad \quad \quad 1
 \end{array}
 \quad
 \begin{array}{c}
 s(x) \times 0 \\
 \swarrow \searrow \\
 0 \quad (x \times 0) + 0 \\
 \quad \quad \quad \curvearrowleft \quad \curvearrowright \\
 \quad \quad \quad 2 \quad \quad 1
 \end{array}
 \quad
 \begin{array}{c}
 s(x) \times 0 \\
 \swarrow \searrow \\
 (x \times 0) + 0 \quad 0 \\
 \quad \quad \quad \curvearrowleft \quad \curvearrowright \\
 \quad \quad \quad 1 \quad \quad 2
 \end{array}$

and subsystem over $\mathcal{F}un(\{1, 2\}) = \{0, +, \times\}$ is $\{1, 2, 3\}$

2 $CR(\{1, 2, 3\}) \iff CR(\emptyset)$ because

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and subsystem over $\mathcal{F}un(\emptyset) = \emptyset$ is \emptyset

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 \begin{array}{c} s(x) \times 0 \\ \swarrow \not\equiv \searrow \\ (x \times 0) + 0 \quad 0 \\ \text{---} \text{---} \text{---} \\ \text{1} \quad \cdot \quad \text{2} \end{array}
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and subsystem over $\mathcal{F}un(\emptyset) = \emptyset$ is \emptyset

using **reduction method**, we show confluence of left-linear TRS \mathcal{R} :

$$\begin{array}{lll} 1: & x + 0 \rightarrow x & 3: & 0 + y \rightarrow y & 5: & s(x) + y \rightarrow s(x + y) \\ 2: & x \times 0 \rightarrow 0 & 4: & s(x) \times 0 \rightarrow 0 & 6: & s(x) \times y \rightarrow (x \times y) + y \end{array}$$

1 $\text{CR}(\mathcal{R}) \iff \text{CR}(\{1, 2, 3\})$ because

Diagram illustrating confluence for rules 1, 2, and 3. The terms are arranged in two rows. The top row terms are $0 + 0$, 0×0 , $s(x) + 0$, $s(x) + 0$, $s(x) \times 0$, and $s(x) \times 0$. The bottom row terms are $0 = 0$, $0 = 0$, $s(x)$, $s(x + 0)$, $s(x + 0)$, $s(x)$, 0 , $(x \times 0) + 0$, $(x \times 0) + 0$, and 0 . Red arrows and numbers indicate the confluence steps: a red arrow labeled '1' connects $s(x)$ and $s(x + 0)$ to $s(x + 0)$; another red arrow labeled '1' connects $s(x + 0)$ and $s(x)$ to $s(x + 0)$; a red arrow labeled '2' connects 0 and $(x \times 0) + 0$ to $(x \times 0) + 0$; and another red arrow labeled '2' connects $(x \times 0) + 0$ and 0 to $(x \times 0) + 0$.

and subsystem over $\mathcal{F}\text{un}(\{1, 2\}) = \{0, +, \times\}$ is $\{1, 2, 3\}$

2 $\text{CR}(\{1, 2, 3\}) \iff \text{CR}(\emptyset)$ because

Diagram illustrating confluence for rule 1. The top term is $0 + 0$ and the bottom term is $0 = 0$. Two arrows point from $0 + 0$ to $0 = 0$.

and subsystem over $\mathcal{F}\text{un}(\emptyset) = \emptyset$ is \emptyset

3 $\text{CR}(\emptyset)$ is trivial,

using **reduction method**, we show confluence of left-linear TRS \mathcal{R} :

$$\begin{array}{lll}
 1: & x + 0 \rightarrow x & 3: & 0 + y \rightarrow y & 5: & s(x) + y \rightarrow s(x + y) \\
 2: & x \times 0 \rightarrow 0 & 4: & s(x) \times 0 \rightarrow 0 & 6: & s(x) \times y \rightarrow (x \times y) + y
 \end{array}$$

1 $CR(\mathcal{R}) \iff CR(\{1, 2, 3\})$ because

and subsystem over $\mathcal{F}un(\{1, 2\}) = \{0, +, \times\}$ is $\{1, 2, 3\}$

2 $CR(\{1, 2, 3\}) \iff CR(\emptyset)$ because

and subsystem over $\mathcal{F}un(\emptyset) = \emptyset$ is \emptyset

3 $CR(\emptyset)$ is trivial, hence $CR(\mathcal{R})$